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A class of orthotropic and transversely isotropic hyperelastic constitutive models based on a polyconvex strain energy function

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Abstract

In the present paper we propose a set of orthotropic and transversely isotropic strain energy functions that (a) are polyconvex, (b) are proved to be coercive and (c) satisfy a priori the condition of the stress-free natural state. These conditions ensure the existence of the global minimizer of the total elastic energy and for this reason are very important in the context of a boundary value problem. The proposed hyperelastic model is represented by a power series with an arbitrary number of terms and corresponding material constants which can easily be evaluated from experimental data. For illustration, the model is fitted to uniaxial tension tests of calendered rubber sheets revealing transverse isotropy with respect to the calendering direction. Thus, a very good agreement with the experimental results is achieved.

Keywords: Orthotropy; Transverse isotropy; Hyperelasticity; Polyconvexity; Coercivity; Incompressibility; Calendered rubber

1. Introduction

A solution of any boundary value problem in nonlinear elasticity depends on the crucial question whether or not there exists a deformation which minimizes in the local or global sense the total elastic energy of the body. This question can be answered positively if the strain energy function of the body (1) is quasiconvex and (2) satisfies some continuity and growth requirements the last one being referred to as coercivity. Indeed, Morrey (1952) (see also Acerbi and Fusco, 1984; Ball and Murat, 1984) showed that these conditions are sufficient for the existence of the global minimizer of the elastic energy. Thereby, the quasiconvexity condition introduced by Morrey ensures that the strain energy function satisfies the Legendre–Hadamard or ellipticity condition. It implies that the acoustic tensor is positive definite such that the speed of displacement waves is always real for any direction of propagation. However, the quasiconvexity condition represents an integral inequality and is very difficult to verify. Instead, a stronger condition

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of polyconvexity has been introduced by Ball (1977). It appears to be easier to handle at least for isotropic materials. Indeed, a great variety of realistic models of nonlinear isotropic materials as for example Ogden, Mooney-Rivlin and neo-Hookean one can be shown to satisfy this condition (Ball, 1977; Ciarlet, 1988). Further, some polyconvex isotropic strain energy functions based on the volumetric-isochoric split have been proposed by Dacorogna (1989) and recently by Hartmann and Neff (2003). On the other hand, St. Venant-Kirchhoff isotropic models based on generalized strain measures are proved to be non-polyconvex (Raoult, 1986; Ciarlet, 1988; Bruhns et al., 2001; Böhlke and Bertram, 2002).

The verification of the polyconvexity condition for many useful anisotropic strain energy functions appears to be a more complicated task. Instead, one can formulate new strain energy functions that are a priori polyconvex. Thus, some polyconvex transversely isotropic and orthotropic energy ansatzes have recently been proposed by Schröder and Neff (2003). However, these ansatzes generally do not satisfy the condition of the stress-free natural state. Thus, a linear combination of these polyconvex terms is required where the coefficients are subjected to an additional constraint. Besides, the question about the coercivity of the proposed anisotropic strain energy functions remains open. Recall, that together with the polyconvexity and continuity the coercivity condition ensures the existence of the global minimizer of the total elastic energy.

Based on the results by Schröder and Neff (2003) we propose in the present paper a set of orthotropic and transversely isotropic strain energy functions that (a) are polyconvex, (b) are proved to be coercive and (c) satisfy a priori the condition of the stress-free natural state. These strain energy functions are represented by a power series with an arbitrary number of terms and corresponding material constants which can easily be evaluated from experimental data. For illustration we match our hyperelastic model to uniaxial tension tests of calendered rubber sheets revealing transverse isotropy with respect to the calendering direction (Diani et al., 2003). Thus, a very good agreement with the experimental results is obtained.

The paper is organized as follows. We begin with some underlying mathematical notations and definitions (Section 2). Then, basic facts about hyperelasticity (Section 3), orthotropic and transversely isotropic material symmetries (Section 4) are recalled. For a given material symmetry the St. Venant-Kirchhoff model represents the simplest hyperelastic formulation and attracts for this reason particular attention. For orthotropic materials it is formulated in Section 5. In Section 6 we recall some known polyconvex anisotropic energy ansatzes which are used in Section 7 to construct a set of orthotropic and transversely isotropic strain energy functions a priori satisfying the condition of the stress-free natural state. These functions are proved to fulfill also the coercivity requirement which we focus on in Section 8. Of particular interest is the special case of incompressible materials considered in Section 9. Finally, our model is very accurately fitted to experimental results on calendered rubber sheets (Section 10).

2. Mathematical preliminaries and notations

Let $\mathbf{Lin}$ be a set of all linear mappings of a three-dimensional vector space $\mathbb{R}^3$ over reals into itself. The elements of $\mathbf{Lin}$ are called second-order tensors. Through the standard operations of sum, multiplication with a scalar and finally the scalar product $\mathbf{Lin}$ represents a finite-dimensional vector space with the inner product. Symmetric, orthogonal and invertible second-order tensors constitute subsets of $\mathbf{Lin}$ defined as follows: $\mathbf{Sym} = \{\mathbf{A} \in \mathbf{Lin} : \mathbf{A} = \mathbf{A}^{\mathrm{T}}\}$, $\mathbf{Orth} = \{\mathbf{Q} \in \mathbf{Lin} : \mathbf{Q} = \mathbf{Q}^{-\mathrm{T}}\}$, $\mathbf{Inv} = \{\mathbf{A} \in \mathbf{Lin} : \det\mathbf{A} \neq 0\}$, where $\det\mathbf{A}$ denotes the determinant of the second-order tensor $\mathbf{A}$. Further, we use the following abbreviations $\operatorname{cof}\mathbf{A} = \mathbf{A}^{-\mathrm{T}}\det\mathbf{A}$ and $\operatorname{adj}\mathbf{A} = \mathbf{A}^{-1}\det\mathbf{A} = (\operatorname{cof}\mathbf{A})^{\mathrm{T}}$, where $\operatorname{cof}\mathbf{A}$ is referred to as cofactor of the tensor $\mathbf{A} \in \mathbf{Inv}$.

Fourth-order tensors in turn form a set $\mathscr{L}\mathbf{in}$ of all linear mappings of $\mathbf{Lin}$ into itself such that

$$\mathbf{B} = \mathscr{D} : \mathbf{A}, \quad \mathbf{B} \in \mathbf{Lin}, \quad \forall \mathbf{A} \in \mathbf{Lin}, \quad \forall \mathscr{D} \in \mathscr{L}\mathbf{in}. \tag{1}$$

They can be constructed from second-order tensors by means of the tensor products "×" and "⊗" defined by

$$(\mathbf{A}\times\mathbf{B}):\mathbf{C}=(\mathbf{B}:\mathbf{C})\mathbf{A},\quad \mathbf{A}\otimes\mathbf{B}:\mathbf{C}=\mathbf{ACB},\quad \forall\mathbf{A},\mathbf{B},\mathbf{C}\in\mathbf{L}\text{in}. \tag{2}$$

The composition of second-order tensors has priority with respect to the tensor products such that $\mathbf{AB}\times\mathbf{CD}=(\mathbf{AB})\times(\mathbf{CD})$ and $\mathbf{AB}\otimes\mathbf{CD}=(\mathbf{AB})\otimes(\mathbf{CD})$. Further, introducing a simple contraction of a fourth-order tensor with second-order ones by

$$(\mathbf{A}\mathscr{D}\mathbf{B}):\mathbf{C}=\mathbf{A}(\mathscr{D}:\mathbf{C})\mathbf{B},\quad \forall\mathscr{D}\in\mathscr{L}\text{in},\quad \forall\mathbf{A},\mathbf{B},\mathbf{C}\in\mathbf{L}\text{in}, \tag{3}$$

one can formulate the following product rules of differentiation (Itskov, 2002)

$$(f\mathbf{A})_{,\mathbf{C}}=\mathbf{A}\times f_{,\mathbf{C}}+f\mathbf{A}_{,\mathbf{C}},\quad (\mathbf{AB})_{,\mathbf{C}}=\mathbf{A}_{,\mathbf{C}}\mathbf{B}+\mathbf{AB}_{,\mathbf{C}},\quad \forall\mathbf{A},\mathbf{B},\mathbf{C}\in\mathbf{L}\text{in}, \tag{4}$$

where f, $\mathbf{A}$ and $\mathbf{B}$ represent a scalar- and two tensor-valued differentiable tensor functions, respectively.

For the functions of several arguments $\mathbf{A}_1,\mathbf{A}_2,\ldots,\mathbf{A}_n$ the following abbreviated notation will be used: $f=\hat{f}(\mathbf{A}_i)(i=1,2,\ldots,n)$.

3. Hyperelastic materials

In the following we will deal with the so-called hyperelastic materials. An elastic material is said to be hyperelastic if its free energy per unit volume of the reference configuration can be represented as a function of the deformation gradient $\mathbf{F}$ by

$$W=W_F(\mathbf{F}). \tag{5}$$

According to the material objectivity condition

$$W_F(\mathbf{QF})=W_F(\mathbf{F}),\quad \forall\mathbf{Q}\in\mathbf{O}\text{rth}. \tag{6}$$

Thus, the strain energy function can be given in terms of the right Cauchy–Green tensor $\mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}$ by

$$W=W_C(\mathbf{C}). \tag{7}$$

A constitutive law can then be written for an unconstrained hyperelastic material by (see e.g. Truesdell and Noll, 1965)

$$\mathbf{S}=2\frac{\partial W}{\partial\mathbf{C}}, \tag{8}$$

where $\mathbf{S}$ represents the second Piola–Kirchhoff stress tensor. The material time derivative of (8) (denoted by a superposed dot) further yields

$$\dot{\mathbf{S}}=\mathscr{C}:\frac{1}{2}\dot{\mathbf{C}}, \tag{9}$$

where

$$\mathscr{C}=2\frac{\partial\mathbf{S}}{\partial\mathbf{C}}=4\frac{\partial^2 W}{\partial\mathbf{C}\partial\mathbf{C}} \tag{10}$$

stands for the so-called tangent (elasticity) tensor of the fourth-order.

For constrained materials the constitutive law is written by

$$\mathbf{S}=2\frac{\partial W}{\partial\mathbf{C}}+q\frac{\partial\gamma}{\partial\mathbf{C}}, \tag{11}$$

where q is an arbitrary scalar and γ represents the so-called constraint function defining by

$$\gamma(\mathbf{C}) = 0 \tag{12}$$

the constraint manifold. The stress rate takes in this case the following form (see e.g. Ogden, 1984)

$$\dot{\mathbf{S}} = \mathscr{C} : \frac{1}{2}\dot{\mathbf{C}} + \dot{q}\frac{\partial \gamma}{\partial \mathbf{C}}, \tag{13}$$

where

$$\mathscr{C} = 4\frac{\partial^2 W}{\partial \mathbf{C}\partial \mathbf{C}} + 2q\frac{\partial^2 \gamma}{\partial \mathbf{C}\partial \mathbf{C}}. \tag{14}$$

4. Orthotropic and transversely isotropic material symmetry

A material is said to be orthotropic if it is characterized by symmetry with respect to three mutually orthogonal planes, by reflections from which material properties remain unchanged. The axes normal to these planes are called principal material directions. The set of all orthogonal mappings that do not violate the material symmetry forms a group referred to as symmetry group. Let $\boldsymbol{l}_i \cdot \boldsymbol{l}_j = \delta_{ij}$ $(i, j = 1, 2, 3)$ be unit base vectors in the principal material directions. Then, the orthotropic symmetry group can be described by means of the so-called structural tensors defined by

$$\mathbf{L}_i = \boldsymbol{l}_i \otimes \boldsymbol{l}_i, \quad i = 1, 2, 3. \tag{15}$$

Transverse isotropy represents a material symmetry with respect to only one selected (principal material) direction. It is characterized by invariance of material properties with respect to rotations about, and reflections from the planes orthogonal or parallel to this direction. The structural tensors can be expressed in this case by

$$\mathbf{L}_1 = \boldsymbol{l}_1 \otimes \boldsymbol{l}_1, \quad \mathbf{L}_2 = \mathbf{L}_3 = \frac{1}{2}(\mathbf{I} - \boldsymbol{l}_1 \otimes \boldsymbol{l}_1), \tag{16}$$

where $\mathbf{I}$ represents the second-order identity tensor and the principal material direction is denoted by the index 1.

With the aid of the structural tensors (15) and (16) the symmetry group can uniquely be defined by

$$\boldsymbol{g} = \{\mathbf{Q} \in \mathbf{O}\mathrm{rth} : \mathbf{Q}\mathbf{L}_i\mathbf{Q}^{\mathrm{T}} = \mathbf{L}_i, \quad i = 1, 2, \ldots, n\}, \tag{17}$$

where n takes the value of 3 for the orthotropic and of 2 for the transversely isotropic symmetry.

The structural tensors (15) and (16) are characterized by the following important properties

$$\sum_i^3 \mathbf{L}_i = \mathbf{I}, \quad \mathbf{L}_i\mathbf{L}_j = \mathbf{0}, \quad \mathrm{tr}\,\mathbf{L}_i = 1, \quad i \neq j; \;\; i, j = 1, 2, \ldots, n. \tag{18}$$

For hyperelastic materials the condition of material symmetry is written in terms of the strain energy function (7) and the symmetry group (17) by

$$W_C(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}) = W_C(\mathbf{C}), \quad \forall \mathbf{Q} \in \boldsymbol{g}. \tag{19}$$

According to the Rychlewski's theorem (see e.g. Zhang and Rychlewski, 1990) this condition is satisfied if and only if the strain energy can be represented by an isotropic tensor function of arguments the list of which additionally includes the structural tensors

$$W = W_{CL}(\mathbf{C}, \mathbf{L}_i), \quad W_{CL}(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{L}_i\mathbf{Q}^{\mathrm{T}}) = W_{CL}(\mathbf{C}, \mathbf{L}_i), \quad i = 1, 2, \ldots, n, \;\; \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}. \tag{20}$$

Indeed, in this case (see also Boehler, 1979; Liu, 1982)

$$W_{CL}(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{L}_i) = W_{CL}(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{L}_i\mathbf{Q}^{\mathrm{T}}) = W_{CL}(\mathbf{C}, \mathbf{L}_i), \quad i = 1, 2, \ldots, n, \ \ \forall \mathbf{Q} \in \boldsymbol{g}. \tag{21}$$

In turn, the isotropic tensor function (20) can be constructed with the aid of the classical invariant theory (see e.g. Smith, 1971). Accordingly, one can write under consideration of (18)

$$W = \widehat{W}(\mathrm{tr}(\mathbf{C}\mathbf{L}_i), \mathrm{tr}(\mathbf{C}^2\mathbf{L}_i), \mathrm{tr}\,\mathbf{C}^3), \quad i = 1, 2, \ldots, n. \tag{22}$$

5. St. Venant-Kirchhoff material

For a given material symmetry the St. Venant-Kirchhoff model represents the simplest hyperelastic formulation and attracts for this reason particular attention. In order to formulate the St. Venant-Kirchhoff model for the orthotropic material symmetry ($n = 3$) we again focus on the representation of the strain energy function (22). First, we rewrite it in terms of the Green–Lagrange strain tensor $\mathbf{E} = 1/2(\mathbf{C} - \mathbf{I})$ by:

$$W = \breve{W}(\mathrm{tr}(\mathbf{E}\mathbf{L}_i), \mathrm{tr}(\mathbf{E}^2\mathbf{L}_i), \mathrm{tr}\,\mathbf{E}^3), \quad i = 1, 2, 3. \tag{23}$$

By virtue of (15) and (18) we can further obtain the following identities

$$\mathrm{tr}(\mathbf{E}^2\mathbf{L}_i) = \mathrm{tr}(\mathbf{E}\mathbf{L}_i\mathbf{E}\mathbf{L}_i) + \mathrm{tr}(\mathbf{E}\mathbf{L}_j\mathbf{E}\mathbf{L}_i) + \mathrm{tr}(\mathbf{E}\mathbf{L}_k\mathbf{E}\mathbf{L}_i), \quad i \neq j \neq k \neq i = 1, 2, 3, \tag{24}$$

$$\mathrm{tr}(\mathbf{E}\mathbf{L}_i\mathbf{E}\mathbf{L}_i) = \mathrm{tr}^2(\mathbf{E}\mathbf{L}_i), \quad i = 1, 2, 3. \tag{25}$$

Thus, one can write

$$W = \breve{\breve{W}}(\mathrm{tr}(\mathbf{E}\mathbf{L}_i), \mathrm{tr}(\mathbf{E}\mathbf{L}_i\mathbf{E}\mathbf{L}_j), \mathrm{tr}\,\mathbf{E}^3), \quad i, j = 1, 2, 3; \ \ j > i. \tag{26}$$

The invariants $\mathrm{tr}(\mathbf{E}\mathbf{L}_i)$ and $\mathrm{tr}(\mathbf{E}\mathbf{L}_i\mathbf{E}\mathbf{L}_j)$ are advantageous as arguments of the strain energy function since they have a clear geometrical interpretation. Indeed, $2\mathrm{tr}(\mathbf{E}\mathbf{L}_i)$ represents the change of the length square in the principal material direction $i (i = 1, 2, 3)$ while $\mathrm{tr}(\mathbf{E}\mathbf{L}_i\mathbf{E}\mathbf{L}_j)$ describes the angle change in the principal material plane $ij (i \neq j)$, that is orthogonal to the principal material direction $k (j \neq k \neq i)$.

Now, we consider a power series expansion of the strain energy function (26) with respect to $\mathbf{E}$:

$$W = W^{(0)} + W^{(1)} + W^{(2)} + \mathrm{O}(\|\mathbf{E}\|^3). \tag{27}$$

The zeroth $W^{(0)}$ and first-order term $W^{(1)}$ should identically vanish in order to satisfy the requirements of the natural state:

$$W|_{\mathbf{E}=\mathbf{0}} = 0, \quad \mathbf{S}|_{\mathbf{E}=\mathbf{0}} = \left.\frac{\partial W}{\partial \mathbf{E}}\right|_{\mathbf{E}=\mathbf{0}} = \mathbf{0}. \tag{28}$$

Hence, we can write

$$W = W^{(2)}(\mathbf{E}) + \mathrm{O}(\|\mathbf{E}\|^3), \tag{29}$$

where the quadratic form (see e.g. Itskov, 2001)

$$W^{(2)}(\mathbf{E}) = \frac{1}{2}\sum_{i,j}^{3} a_{ij}\mathrm{tr}(\mathbf{E}\mathbf{L}_i)\mathrm{tr}(\mathbf{E}\mathbf{L}_j) + \sum_{i,j\neq i}^{3} G_{ij}\mathrm{tr}(\mathbf{E}\mathbf{L}_i\mathbf{E}\mathbf{L}_j) \tag{30}$$

is regarded as the St. Venant-Kirchhoff model. It is given in terms of the so-called engineering elastic constants

$$\begin{aligned} &E_i (i=1,2,3): && \text{Young's moduli,}\\ &G_{ij}=G_{ji}\ (i\neq j=1,2,3): && \text{Lamé's shear moduli,}\\ &v_{ij}=v_{ji}\frac{E_j}{E_i}\ (i\neq j=1,2,3): && \text{Poisson's ratios} \end{aligned} \tag{31}$$

referred to the principal material directions and planes, respectively. Thereby, the material constants a_{ij} $(i,j=1,2,3)$ are expressed by (see e.g. Jones, 1975)

$$\begin{aligned} &a_{ii}=E_i\frac{1-v_{jk}v_{kj}}{\Delta},\quad a_{ij}=a_{ji}=E_i\frac{v_{ij}+v_{kj}v_{ik}}{\Delta},\quad i\neq j\neq k\neq i,\ \ i,j=1,2,3,\\ &\Delta=1-v_{12}v_{21}-v_{23}v_{32}-v_{31}v_{13}-2v_{21}v_{32}v_{13}. \end{aligned} \tag{32}$$

For incompressible materials a_{ij} $(i,j=1,2,3)$ take the form (see e.g. Itskov and Aksel, 2002)

$$a_{ii}=\frac{1}{3D}\left(\frac{2}{E_j}+\frac{2}{E_k}-\frac{1}{E_i}\right),\quad a_{ij}=\frac{1}{6D}\left(\frac{1}{E_i}+\frac{1}{E_j}-\frac{5}{E_k}\right),\quad i\neq j\neq k\neq i,\ \ i,j=1,2,3 \tag{33}$$

with the abbreviation

$$D=\frac{3}{4}\left(\frac{2}{E_1E_2}+\frac{2}{E_2E_3}+\frac{2}{E_3E_1}-\frac{1}{E_1^2}-\frac{1}{E_2^2}-\frac{1}{E_3^2}\right). \tag{34}$$

In the case of transversely isotropic material symmetry, all directions orthogonal to the principal material one become equivalent. This can be taken into account by setting

$$E_2=E_3,\quad v_{12}=v_{13}(v_{21}=v_{31}),\quad G_{12}=G_{31},\quad G_{23}=\frac{E_2}{2(1+v_{23})}. \tag{35}$$

Hence, for transversely isotropic incompressible materials

$$a_{11}=\frac{4}{9}E_1,\quad a_{22}=a_{33}=\frac{1}{3D}\left(\frac{2}{E_1}+\frac{1}{E_2}\right),\quad a_{23}=\frac{1}{6D}\left(\frac{2}{E_2}-\frac{5}{E_1}\right),\quad a_{12}=a_{13}=-\frac{2}{9}E_1, \tag{36}$$

where

$$D=\frac{3}{4E_1}\left(\frac{4}{E_2}-\frac{1}{E_1}\right). \tag{37}$$

The constitutive relations and elastic moduli corresponding to the strain energy function (30) are of the form

$$\mathbf{S}=\frac{\partial W^{(2)}}{\partial\mathbf{E}}=\sum_{i,j}^{3}a_{ij}\mathrm{tr}(\mathbf{E}\mathbf{L}_i)\mathbf{L}_j+\sum_{i,j\neq i}^{3}2G_{ij}\mathbf{L}_i\mathbf{E}\mathbf{L}_j, \tag{38}$$

$$\mathscr{C}=\frac{\partial^2 W^{(2)}}{\partial\mathbf{E}\partial\mathbf{E}}=\sum_{i,j}^{3}a_{ij}\mathbf{L}_i\times\mathbf{L}_j+\sum_{i,j\neq i}^{3}2G_{ij}(\mathbf{L}_i\otimes\mathbf{L}_j)^{\mathrm{S}}, \tag{39}$$

where $(\bullet)^{\mathrm{S}}$ denotes a symmetrization operation on fourth-order tensors defined by (Itskov, 2002)

$$\mathscr{D}^{\mathrm{S}}:\mathbf{A}=\mathscr{D}:\frac{1}{2}(\mathbf{A}+\mathbf{A})^{\mathrm{T}},\quad \forall\mathbf{A}\in\mathbf{L}\mathrm{in},\quad \forall\mathscr{D}\in\mathscr{L}\mathrm{in}. \tag{40}$$

6. Polyconvex strain energy functions

A strain energy function $W_F(\mathbf{F}) : \mathrm{Inv} \to \mathbb{R}$ is said to be polyconvex (Ball, 1977) if and only if there exists a convex function $\widehat{W}(\mathbf{F}, \mathrm{adj}\,\mathbf{F}, \det\mathbf{F}) : (\mathrm{Inv}, \mathrm{Inv}, \mathbb{R}^+) \to \mathbb{R}$ such that

$$W_F(\mathbf{F}) = \widehat{W}(\mathbf{F}, \mathrm{adj}\,\mathbf{F}, \det\mathbf{F}). \tag{41}$$

Note, that $\mathbf{F}$, $\mathrm{adj}\,\mathbf{F}$ and $\det\mathbf{F}$ describe deformations of line, surface and volume elements, respectively.

There are some powerful tools enabling to construct a great variety of polyconvex strain functions. One of them is the additive representation (see e.g. Schröder and Neff, 2003)

$$W_F(\mathbf{F}) = \widehat{W}_1(\mathbf{F}) + \widehat{W}_2(\mathrm{adj}\,\mathbf{F}) + \widehat{W}_3(\det\mathbf{F}). \tag{42}$$

If each of $\widehat{W}_i (i = 1, 2, 3)$ is convex then the strain energy function $W_F(\mathbf{F})$ (42) is polyconvex. Thus, the problem of the construction of polyconvex functions can be reduced to the formulation of convex ones. Thereby, the following well-known statement is useful:

Let $\varphi(\mathbf{F}) : \mathrm{Inv} \to \mathbb{R}$ be convex and non-negative. Then, the function $\varphi^p(\mathbf{F})$ is convex for $p \geqslant 1$.

Finally, we recall some important results (see Schröder and Neff, 2003 for details) concerning convexity of functions of the right Cauchy–Green tensor $\mathbf{C}$. Let $\mathbf{D} \in \mathbf{L}\mathrm{in}$ be an arbitrary positive-definite second-order tensor. Then, the functions

$$\mathrm{tr}(\mathbf{CD}), \quad \mathrm{tr}[(\mathrm{cof}\,\mathbf{C})\mathbf{D}] \tag{43}$$

are convex with respect to $\mathbf{F}$ and $\mathrm{adj}\,\mathbf{F}$, respectively (Schröder and Neff, 2003, p. 415). In contrast, the functions

$$\mathrm{tr}(\mathbf{C}^2\mathbf{D}), \quad \mathrm{tr}[(\mathrm{cof}\,\mathbf{C})^2\mathbf{D}] \tag{44}$$

are not convex. In the next section the above results will be used to construct orthotropic and transversely isotropic polyconvex strain energy functions a priori satisfying the condition of the stress-free natural state $(28)_2$.

7. A set of orthotropic and transversely isotropic polyconvex strain energy functions

The invariants $\mathrm{tr}(\mathbf{C}^2\mathbf{L}_i), (i = 1, 2, \ldots, n)$ and $\mathrm{tr}\,\mathbf{C}^3$ appearing in the general representation of the strain energy function (22) are non-convex with respect to $\mathbf{F}$, $\mathrm{adj}\,\mathbf{F}$ or $\det\mathbf{F}$ and should first be expressed in terms of convex ones. To this end, we apply the Cayley–Hamilton theorem written for the right Cauchy–Green tensor by

$$\mathbf{C}^3 - \mathrm{I}_\mathbf{C}\mathbf{C}^2 + \mathrm{II}_\mathbf{C}\mathbf{C} - \mathrm{III}_\mathbf{C}\mathbf{I} = \mathbf{0}, \tag{45}$$

where the scalar coefficients

$$\mathrm{I}_\mathbf{C} = \mathrm{tr}\,\mathbf{C}, \quad \mathrm{II}_\mathbf{C} = \frac{1}{2}[(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}\,\mathbf{C}^2], \quad \mathrm{III}_\mathbf{C} = \det\mathbf{C} \tag{46}$$

represent the principal invariants of $\mathbf{C}$. Multiplying (45) with $\mathbf{C}^{-1}$ we further obtain

$$\mathbf{C}^2 - \mathrm{I}_\mathbf{C}\mathbf{C} + \mathrm{II}_\mathbf{C}\mathbf{I} - \mathrm{cof}\,\mathbf{C} = \mathbf{0}. \tag{47}$$

Thus, one gains the following alternative representation of the strain energy function (22)

$$W = \overline{W}(I_i, J_i, \mathrm{III}_\mathbf{C}), \quad i = 1, 2, \ldots, n, \tag{48}$$

in terms of the invariants

$$I_i = \mathrm{tr}(\mathbf{CL}_i), \quad J_i = \mathrm{tr}[(\mathrm{cof}\,\mathbf{C})\mathbf{L}_i], \quad i = 1, 2, \ldots, n \tag{49}$$

and $\mathrm{III}_{\mathbf{C}}$ convex with respect to $\mathbf{F}$, adj $\mathbf{F}$ and det$\mathbf{F}$, respectively. Note also, that in view of $(18)_1$, (46) and (47)

$$\sum_{i}^{3} I_i = \mathrm{I}_{\mathbf{C}}, \quad \sum_{i}^{3} J_i = \mathrm{tr}(\mathrm{cof}\,\mathbf{C}) = \mathrm{II}_{\mathbf{C}}. \tag{50}$$

Due to positive semi-definiteness of the structural tensors (15) and (16)

$$I_i > 0, \quad J_i > 0, \quad i = 1, 2, \ldots, n. \tag{51}$$

Further, we introduce the following linear combinations of the invariants (49)

$$\widetilde{I}_r = \sum_{i}^{n} w_i^{(r)} I_i, \quad \widetilde{J}_r = \sum_{i}^{n} w_i^{(r)} J_i, \quad r = 1, 2, \ldots \tag{52}$$

They are likewise convex and non-negative if the weight factors of the principal material directions $w_i^{(r)}$ are non-negative:

$$w_i^{(r)} \geqslant 0, \quad i = 1, 2, \ldots, n; \;\; r = 1, 2, \ldots \tag{53}$$

$\widetilde{I}_r$ and $\widetilde{J}_r$ (52) represent generalized invariants and can alternatively be expressed by

$$\widetilde{I}_r = \mathrm{tr}(\mathbf{C}\widetilde{\mathbf{L}}_r), \quad \widetilde{J}_r = \mathrm{tr}[(\mathrm{cof}\,\mathbf{C})\widetilde{\mathbf{L}}_r], \;\; r = 1, 2, \ldots \tag{54}$$

in terms of the generalized structural tensors defined by

$$\widetilde{\mathbf{L}}_r = \sum_{i}^{n} w_i^{(r)} \mathbf{L}_i, \quad r = 1, 2, \ldots \tag{55}$$

To ensure the property $(18)_3$

$$\mathrm{tr}\widetilde{\mathbf{L}}_r = 1, \quad r = 1, 2, \ldots \tag{56}$$

we further require that

$$\sum_{i}^{n} w_i^{(r)} = 1, \quad r = 1, 2, \ldots \tag{57}$$

Now, let us consider a strain energy function of the form

$$W = \frac{1}{4} \sum_{r}^{s} \mu_r \left[\frac{1}{\alpha_r} (\widetilde{I}_r^{\alpha_r} - 1) + \frac{1}{\beta_r} (\widetilde{J}_r^{\beta_r} - 1) + \frac{1}{\gamma_r} (\mathrm{III}_{\mathbf{C}}^{-\gamma_r} - 1) \right], \tag{58}$$

where μ_r, α_r, β_r and $\gamma_r (r = 1, 2, \ldots, s)$ represent material constants. In view of (42) the function (58) is polyconvex if

$$\mu_r \geqslant 0, \quad \alpha_r \geqslant 1, \quad \beta_r \geqslant 1, \quad \gamma_r \geqslant -\frac{1}{2}, \;\; r = 1, 2, \ldots, s. \tag{59}$$

It is seen that the polyconvexity domain includes zero values of $\gamma_r (r = 1, 2, \ldots, s)$. In this case the last term in (58) transforms to

$$\lim_{\gamma_r \to 0} \frac{1}{\gamma_r} (\mathrm{III}_{\mathbf{C}}^{-\gamma_r} - 1) = -\ln \mathrm{III}_{\mathbf{C}}. \tag{60}$$

The strain energy function (58) a priori satisfies the conditions of the energy and stress free natural state (28). Indeed, the constitutive relations resulting from the strain energy function (58) are of the form

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}} = \frac{1}{2} \sum_{r}^{s} \mu_r \left[\widetilde{I}_r^{\alpha_r - 1} \widetilde{\mathbf{L}}_r - \widetilde{J}_r^{\beta_r - 1} \mathrm{III}_{\mathbf{C}} \mathbf{C}^{-1} \widetilde{\mathbf{L}}_r \mathbf{C}^{-1} + \left(\widetilde{J}_r^{\beta_r} - \mathrm{III}_{\mathbf{C}}^{-\gamma_r} \right) \mathbf{C}^{-1} \right]. \tag{61}$$

For the natural state ($\mathbf{C}=\mathbf{I}$) we obtain under consideration of (54) and (56)

$$W|_{\mathbf{C}=\mathbf{I}}=0,\quad \mathbf{S}|_{\mathbf{C}=\mathbf{I}}=\mathbf{0}. \tag{62}$$

The further differentiation of (61) with respect to $\mathbf{C}$ yields the tangent moduli:

$$\begin{aligned}\mathscr{C} &= 2\frac{\partial \mathbf{S}}{\partial \mathbf{C}}\\ &= \sum_r^s \mu_r\Big\{(\alpha_r-1)\widetilde{I}_r^{\alpha_r-2}\widetilde{\mathbf{L}}_r\times\widetilde{\mathbf{L}}_r+(\beta_r-1)\widetilde{J}_r^{\beta_r-2}\mathrm{III}_{\mathbf{C}}^2\Big(\mathbf{C}^{-1}\widetilde{\mathbf{L}}_r\mathbf{C}^{-1}\times\mathbf{C}^{-1}\widetilde{\mathbf{L}}_r\mathbf{C}^{-1}\Big)\\ &\quad+\Big(\beta_r\widetilde{J}_r^{\beta_r}+\gamma_r\mathrm{III}_{\mathbf{C}}^{-\gamma_r}\Big)\mathbf{C}^{-1}\times\mathbf{C}^{-1}-\beta_r\widetilde{J}_r^{\beta_r-1}\mathrm{III}_{\mathbf{C}}\Big(\mathbf{C}^{-1}\times\mathbf{C}^{-1}\widetilde{\mathbf{L}}_r\mathbf{C}^{-1}+\mathbf{C}^{-1}\widetilde{\mathbf{L}}_r\mathbf{C}^{-1}\times\mathbf{C}^{-1}\Big)\\ &\quad-\Big(\widetilde{J}_r^{\beta_r}-\mathrm{III}_{\mathbf{C}}^{-\gamma_r}\Big)\big(\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\big)^{\mathrm{S}}+\widetilde{J}_r^{\beta_r-1}\mathrm{III}_{\mathbf{C}}\Big(\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\widetilde{\mathbf{L}}_r\mathbf{C}^{-1}+\mathbf{C}^{-1}\widetilde{\mathbf{L}}_r\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\Big)^{\mathrm{S}}\Big\}.\end{aligned} \tag{63}$$

Any hyperelastic model has to recover all material constants specific for a given material symmetry. Thus, expanded around the zero strain, zero stress state, the elasticity tensor (63) has to coincide with those one of the St. Venant-Kirchhoff model (39):

$$\mathscr{C}|_{\mathbf{C}=\mathbf{I}}=\frac{\partial^2 W^{(2)}}{\partial\mathbf{E}\partial\mathbf{E}}. \tag{64}$$

Comparing the representation

$$\mathscr{C}|_{\mathbf{C}=\mathbf{I}}=\sum_r^s\mu_r\Big\{(\alpha_r+\beta_r-2)\widetilde{\mathbf{L}}_r\times\widetilde{\mathbf{L}}_r+(\beta_r+\gamma_r)\mathbf{I}\times\mathbf{I}-\beta_r\Big(\mathbf{I}\times\widetilde{\mathbf{L}}_r+\widetilde{\mathbf{L}}_r\times\mathbf{I}\Big)+\Big(\mathbf{I}\otimes\widetilde{\mathbf{L}}_r+\widetilde{\mathbf{L}}_r\otimes\mathbf{I}\Big)^{\mathrm{S}}\Big\} \tag{65}$$

with (39) and taking into account (55) and (57) we obtain for the orthotropic material symmetry

$$\begin{aligned}a_{ij}&=\sum_r^s\mu_r\Big[(\alpha_r+\beta_r-2)w_i^{(r)}w_j^{(r)}+\beta_r\Big(1-w_i^{(r)}-w_j^{(r)}\Big)+\gamma_r+2\delta_{ij}w_i^{(r)}\Big],\quad i,j=1,2,3,\\ 2G_{ij}&=\sum_r^s\mu_r\Big(w_i^{(r)}+w_j^{(r)}\Big),\quad i\neq j,\ \ i,j=1,2,3.\end{aligned} \tag{66}$$

Of particular interest is the special case of isotropy which can be specified by setting

$$w_i^{(r)}=\frac{1}{3}\quad i=1,2,3;\ \ r=1,2,\ldots,s. \tag{67}$$

Thereby, the hyperelastic model (58) reduces to the generalized Mooney-Rivlin material

$$W=\frac{1}{4}\sum_r^s\mu_r\left\{\frac{1}{\alpha_r}\left[\left(\frac{\mathrm{I}_{\mathbf{C}}}{3}\right)^{\alpha_r}-1\right]+\frac{1}{\beta_r}\left[\left(\frac{\mathrm{II}_{\mathbf{C}}}{3}\right)^{\beta_r}-1\right]+\frac{1}{\gamma_r}\left(\mathrm{III}_{\mathbf{C}}^{-\gamma_r}-1\right)\right\}. \tag{68}$$

8. Coercivity

The polyconvexity alone is not sufficient for the existence of the global minimizer of the total elastic energy of the body. In addition, the strain energy function should satisfy a growth condition referred to as coercivity (see Ball, 1977; Ciarlet, 1988). The coercivity condition can be formulated as follows. There exist some $c_0>0$, $p\geqslant 1$, $q\geqslant\frac{3}{4}$ and c_1 such that (Müller et al., 1994):

$$W_C(\mathbf{C})\geqslant c_0(\mathrm{I}_{\mathbf{C}}^p+\mathrm{II}_{\mathbf{C}}^q)-c_1,\ \ \forall\mathbf{C}\in\mathbf{S}\mathrm{ym}^+, \tag{69}$$

where $\mathbf{S}\mathrm{ym}^+$ denotes the set of all symmetric second-order tensors with the positive determinant.

For the proof of the coercivity of the strain energy function (58) we set

$$p = q = 1. \tag{70}$$

We also assume that

$$\gamma_r \geqslant 0, \quad r = 1, 2, \ldots, s \tag{71}$$

and for each principal material direction $i (i = 1, 2, \ldots, n)$ not all weight factors $w_i^{(r)} (r = 1, 2, \ldots, s)$ vanish. Hence,

$$\exists r_i : w_i^{(r_i)} \neq 0, \ \ \forall i = 1, 2, \ldots, n. \tag{72}$$

Thus, in view of the restrictions (51), (53) and (71) and under consideration of (58) we can write

$$W \geqslant \frac{1}{4} \sum_i^n \frac{\mu_{r_i}}{\alpha_{r_i}} \left(w_i^{(r_i)} I_i \right)^{\alpha_{r_i}} + \frac{1}{4} \sum_i^n \frac{\mu_{r_i}}{\beta_{r_i}} \left(w_i^{(r_i)} J_i \right)^{\beta_{r_i}} - \frac{1}{4} \sum_r^s \mu_r \left(\alpha_r^{-1} + \beta_r^{-1} + \gamma_r^{-1} \right). \tag{73}$$

Let

$$w_\alpha = \frac{1}{4} \min_{i=1,2,\ldots,n} \left[\frac{\mu_{r_i}}{\alpha_{r_i}} \left(w_i^{(r_i)} \right)^{\alpha_{r_i}} \right]. \tag{74}$$

Then, one can write for the first sum in (73)

$$\frac{1}{4} \sum_i^n \frac{\mu_{r_i}}{\alpha_{r_i}} \left(w_i^{(r_i)} I_i \right)^{\alpha_{r_i}} \geqslant w_\alpha \sum_i^n I_i^{\alpha_{r_i}}. \tag{75}$$

Recall, that $I_i > 0 \ (i = 1, 2, \ldots, n)$ in view of (51) while $\alpha_{r_i} \geqslant 1 \ (i = 1, 2, \ldots, n)$ according to (59).

Further, let

$$I_i < 1, \ \ i = 1, 2, \ldots, k, \quad I_i \geqslant 1, \ \ i = k+1, k+2, \ldots, n \tag{76}$$

for some $k (0 \leqslant k \leqslant n)$. Then, we obtain

$$w_\alpha \sum_i^k I_i^{\alpha_{r_i}} > 0 > w_\alpha \left(\sum_i^k I_i - k \right), \quad w_\alpha \sum_{i=k+1}^n I_i^{\alpha_{r_i}} \geqslant w_\alpha \sum_{i=k+1}^n I_i. \tag{77}$$

Combining these results yields under consideration of (15), (16), (49) and (50)

$$w_\alpha \sum_{i=1}^n I_i^{\alpha_{r_i}} \geqslant w_\alpha \left(\sum_i^n I_i - k \right) \geqslant w_\alpha \left(\sum_i^n I_i - n \right) \geqslant w_\alpha \left(\frac{\mathrm{I_C}}{2} - n \right). \tag{78}$$

Thus, in view of (75)

$$\frac{1}{4} \sum_i^n \frac{\mu_{r_i}}{\alpha_{r_i}} \left(w_i^{(r_i)} I_i \right)^{\alpha_{r_i}} \geqslant w_\alpha \left(\frac{\mathrm{I_C}}{2} - n \right). \tag{79}$$

For the second sum in (73) we obtain by using a similar procedure

$$\frac{1}{4} \sum_i^n \frac{\mu_{r_i}}{\beta_{r_i}} \left(w_i^{(r_i)} J_i \right)^{\beta_{r_i}} \geqslant w_\beta \left(\frac{\mathrm{II_C}}{2} - n \right), \tag{80}$$

where

$$w_\beta = \frac{1}{4} \min_{i=1,2,\ldots,n} \left[\frac{\mu_{r_i}}{\beta_{r_i}} \left(w_i^{(r_i)} \right)^{\beta_{r_i}} \right]. \tag{81}$$

Under consideration of (73) and with the aid of the abbreviation $w = \frac{1}{2}\min\left(w_\alpha, w_\beta\right)$ this finally leads to the following inequality

$$W \geqslant w(\mathrm{I}_{\mathbf{C}} + \mathrm{II}_{\mathbf{C}}) - \left(w_\alpha + w_\beta\right)n - \frac{1}{4}\sum_r^s \mu_r\left(\alpha_r^{-1} + \beta_r^{-1} + \gamma_r^{-1}\right), \tag{82}$$

which immediately implies (69) and (70), where $c_1 = \left(w_\alpha + w_\beta\right)n + \frac{1}{4}\sum_r^s \mu_r\left(\alpha_r^{-1} + \beta_r^{-1} + \gamma_r^{-1}\right)$ and $c_0 = w$.

9. Special case: incompressible materials

For incompressible materials characterized by the condition

$$\mathrm{III}_{\mathbf{C}} = 1 \tag{83}$$

the strain energy function (58) can be given by

$$W = \frac{1}{4}\sum_r^s \mu_r\left[\frac{1}{\alpha_r}\left(\tilde{I}_r^{\alpha_r} - 1\right) + \frac{1}{\beta_r}\left(\tilde{K}_r^{\beta_r} - 1\right)\right], \tag{84}$$

where

$$\tilde{K}_r = \mathrm{tr}\left(\mathbf{C}^{-1}\tilde{\mathbf{L}}_r\right). \tag{85}$$

The representation (84) is not, however, unique. Indeed, let us consider the functions of the form (see e.g. Casey, 1995)

$$\psi(\mathbf{C}) = W + \varsigma(\mathbf{C})\gamma(\mathbf{C}), \tag{86}$$

where ς is an *arbitrary* continuously differentiable function of the right Cauchy–Green tensor $\mathbf{C}$. For incompressible materials the constraint function can further be given by

$$\gamma(\mathbf{C}) = \mathrm{III}_{\mathbf{C}}^{1/3} - 1. \tag{87}$$

Thus, for all $\varsigma(\mathbf{C})$ the strain energy function ψ (86) coincides with the original one W (84) within the constraint manifold defined by the incompressibility condition (83).

Now, replacing W by ψ (86) in (11) and (14) we obtain under consideration of (83) and (87)

$$\mathbf{S} = 2\frac{\partial\psi}{\partial\mathbf{C}} + q\frac{\partial\gamma}{\partial\mathbf{C}} = \frac{1}{2}\sum_r^s \mu_r\left[\tilde{I}_r^{\alpha_r - 1}\tilde{\mathbf{L}}_r - \tilde{K}_r^{\beta_r - 1}\mathbf{C}^{-1}\tilde{\mathbf{L}}_r\mathbf{C}^{-1}\right] + \frac{1}{3}(q + 2\varsigma)\mathbf{C}^{-1}, \tag{88}$$

$$\begin{aligned}\mathscr{C} &= 4\frac{\partial^2\psi}{\partial\mathbf{C}\partial\mathbf{C}} + 2q\frac{\partial^2\gamma}{\partial\mathbf{C}\partial\mathbf{C}} \\ &= \sum_r^s \mu_r\Big\{(\alpha_r - 1)\tilde{I}_r^{\alpha_r - 2}\tilde{\mathbf{L}}_r \times \tilde{\mathbf{L}}_r + (\beta_r - 1)\tilde{K}_r^{\beta_r - 2}\mathbf{C}^{-1}\tilde{\mathbf{L}}_r\mathbf{C}^{-1} \times \mathbf{C}^{-1}\tilde{\mathbf{L}}_r\mathbf{C}^{-1} \\ &\quad + \tilde{K}_r^{\beta_r - 1}\left(\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\tilde{\mathbf{L}}_r\mathbf{C}^{-1} + \mathbf{C}^{-1}\tilde{\mathbf{L}}_r\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right)^{\mathrm{S}}\Big\} + \frac{4}{3}\frac{\partial\varsigma}{\partial\mathbf{C}} \times \mathbf{C}^{-1} + \frac{4}{3}\mathbf{C}^{-1} \times \frac{\partial\varsigma}{\partial\mathbf{C}} \\ &\quad + \frac{2}{3}(q + 2\varsigma)\left[\frac{1}{3}\mathbf{C}^{-1} \times \mathbf{C}^{-1} - \left(\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right)^{\mathrm{S}}\right].\end{aligned} \tag{89}$$

It is seen that due to arbitrariness of q the value of the function ς influences neither the stress (88) nor the tangent tensor (89). It is not, however, the case for the partial derivative of this function with respect to $\mathbf{C}$

appearing in the tangent tensor (89). Since the terms with $\partial\varsigma/\partial\mathbf{C}$ result in no additional stress power, this derivative cannot generally be determined unless an additional assumption concerning the zero-energy eigenform of the tangent moduli is met. Thus, these terms only indicate the ambiguity of the tangent tensor of a constrained material. However, for a juxtaposition of the tangent tensors (39) and (89) a unique form of the last one is required. Indeed, this juxtaposition is necessary to express the elastic constants (33) in terms of the material parameters μ_r, α_r, β_r $(r = 1, 2, \ldots, s)$ describing the strain energy function (84). Since the elastic constants (33) have been derived under the reasonable assumption that the zero-energy eigenform of the tangent tensor is purely volumetric (see Itskov and Aksel, 2002) we set

$$\mathscr{C} : \mathbf{C} = \mathbf{0}. \tag{90}$$

After some algebra this leads to the following expression

$$\frac{\partial\varsigma}{\partial\mathbf{C}} = \frac{1}{6}\left(\mathbf{C} : \frac{\partial^2 W}{\partial\mathbf{C}\partial\mathbf{C}} : \mathbf{C}\right)\mathbf{C}^{-1} - \frac{\partial^2 W}{\partial\mathbf{C}\partial\mathbf{C}} : \mathbf{C}. \tag{91}$$

It is seen that ignoring the function $\varsigma(\mathbf{C})$ in (86) and setting thus $\partial\varsigma/\partial\mathbf{C} = \mathbf{0}$ in (89) would lead to the violation of the assumption (90) necessary for the comparison of the tangent tensors (39) and (89).

In view of $(62)_2$ and under consideration of (88) $q + 2\varsigma = 0$ in the stress-free natural state. Thus, we obtain

$$\mathscr{C}|_{\mathbf{C}=\mathbf{I}} = \sum_r^s \mu_r \left\{ (\alpha_r + \beta_r - 2)\tilde{\mathbf{L}}_r \times \widetilde{\mathbf{L}}_r + \left(\mathbf{I} \otimes \tilde{\mathbf{L}}_r + \widetilde{\mathbf{L}}_r \otimes \mathbf{I}\right)^{\mathrm{S}} + \frac{1}{3}(\alpha_r + \beta_r)\left(\frac{1}{3}\mathbf{I} \times \mathbf{I} - \tilde{\mathbf{L}}_r \times \mathbf{I} - \mathbf{I} \times \widetilde{\mathbf{L}}_r\right)\right\}. \tag{92}$$

Comparing this tensor with (39) yields

$$\begin{aligned} a_{ij} &= \sum_r^s \mu_r \left[(\alpha_r + \beta_r - 2) w_i^{(r)} w_j^{(r)} + \frac{1}{3}(\alpha_r + \beta_r)\left(\frac{1}{3} - w_i^{(r)} - w_j^{(r)}\right) + 2\delta_{ij} w_i^{(r)} \right], \quad i, j = 1, 2, 3, \\ 2G_{ij} &= \sum_r^s \mu_r \left(w_i^{(r)} + w_j^{(r)}\right), \quad i \neq j, \;\; i, j = 1, 2, 3, \end{aligned} \tag{93}$$

where a_{ij} $(i, j = 1, 2, 3)$ are given by (33).

Of particular interest for the following comparison with experimental data is the special case where the principal material directions and the principal axes of the right Cauchy–Green tensor coincide. In this case,

$$\mathbf{C} = \sum_i^3 \lambda_i^2 \boldsymbol{l}_i \otimes \boldsymbol{l}_i \tag{94}$$

where $\lambda_i (i = 1, 2, 3)$ denote principal stretches. Thus, the strain energy function (84) takes the form

$$W = \frac{1}{4}\sum_r^s \mu_r \left\{ \frac{1}{\alpha_r}\left[\left(w_1^{(r)}\lambda_1^2 + w_2^{(r)}\lambda_2^2 + w_3^{(r)}\lambda_3^2\right)^{\alpha_r} - 1\right] + \frac{1}{\beta_r}\left[\left(w_1^{(r)}\lambda_1^{-2} + w_2^{(r)}\lambda_2^{-2} + w_3^{(r)}\lambda_3^{-2}\right)^{\beta_r} - 1\right]\right\}, \tag{95}$$

where the principal stretches are connected by the incompressibility condition (83)

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{96}$$

For the Cauchy stresses we further obtain

$$\sigma_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} + p$$
$$= \frac{1}{2}\lambda_i^2 \sum_r^s \mu_r w_i^{(r)} \left(w_1^{(r)} \lambda_1^2 + w_2^{(r)} \lambda_2^2 + w_3^{(r)} \lambda_3^2 \right)^{\alpha_r - 1} - \frac{1}{2}\lambda_i^{-2} \sum_r^s \mu_r w_i^{(r)} \left(w_1^{(r)} \lambda_1^{-2} + w_2^{(r)} \lambda_2^{-2} + w_3^{(r)} \lambda_3^{-2} \right)^{\beta_r - 1} + p,$$
$$i = 1, 2, 3, \tag{97}$$

where p represents the hydrostatic pressure.

10. Numerical example: parameter identification

In this section we focus on the numerical identification of the material parameters included in our hyperelastic model. To this end, recently published experimental data by Diani et al. (2003) on calendered rubber sheets appear to be very suitable. In these experiments an anisotropy and in particular transverse isotropy with respect to the calendering direction has been observed. The anisotropy becomes especially apparent in samples made of rubber filled with silica particles. In this case, the difference between stresses by uniaxial tension in calendering and transverse directions can reach 45% (see Fig. 1).

In the case of the transversely isotropic material symmetry ($n = 2$) all directions orthogonal to the principal one are equivalent. Hence, we can set according to (57)

$$w_2^{(r)} = w_3^{(r)} = \frac{1}{2}\left(1 - w_1^{(r)}\right), r = 1, 2, \ldots, s, \tag{98}$$

where the principal material direction is denoted by index "1".

First, we consider the uniaxial loading in the calendering direction. In this case, one can set in view of the incompressibility condition (96): $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = \lambda^{-1/2}$. For the so-called nominal stresses measured in the experiment

$$t_i = \frac{\sigma_{ii}}{\lambda_i} = \frac{\partial W}{\partial \lambda_i} + p\lambda_i^{-1}, \quad i = 1, 2, 3 \tag{99}$$

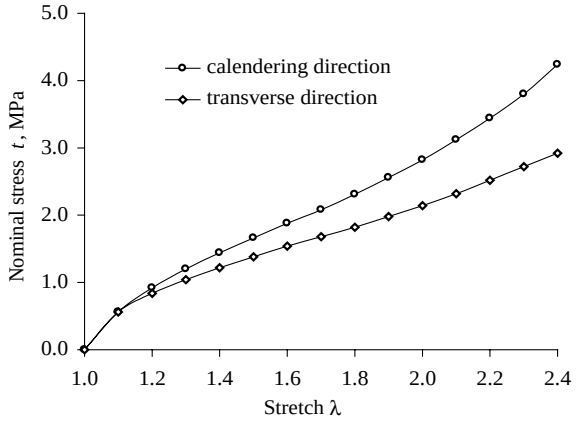

Fig. 1. Experimental results by Diani et al. (2003) on uniaxial tension of calendered rubber filled with silica particles: nominal (1st Piola–Kirchhoff) stresses versus principal stretch.

we thus obtain under consideration of (97) and (98)

$$t_1(\lambda) = \frac{1}{4}\sum_r^s \mu_r \Big\{ \Big[2w_1^{(r)}\lambda - \Big(1 - w_1^{(r)}\Big)\lambda^{-2}\Big]\Big[w_1^{(r)}\lambda^2 + \Big(1 - w_1^{(r)}\Big)\lambda^{-1}\Big]^{\alpha_r - 1} - \Big[2w_1^{(r)}\lambda^{-3} - \Big(1 - w_1^{(r)}\Big)\Big]\Big[w_1^{(r)}\lambda^{-2} + \Big(1 - w_1^{(r)}\Big)\lambda\Big]^{\beta_r - 1}\Big\}, \quad t_2 = t_3 = 0. \tag{100}$$

Further, by the uniaxial loading in the transverse direction $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$. Thus, the nominal stresses take the form

$$t_1(\lambda_1,\lambda_2) = \frac{1}{4}\sum_r^s \mu_r \Big\{ \Big[2w_1^{(r)}\lambda_1 - \Big(1 - w_1^{(r)}\Big)\lambda_1^{-3}\lambda_2^{-2}\Big]\Big[w_1^{(r)}\lambda_1^2 + \Big(1 - w_1^{(r)}\Big)\Big(\frac{\lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}}{2}\Big)\Big]^{\alpha_r - 1} - \Big[2w_1^{(r)}\lambda_1^{-3} - \Big(1 - w_1^{(r)}\Big)\lambda_1\lambda_2^2\Big]\Big[w_1^{(r)}\lambda_1^{-2} + \Big(1 - w_1^{(r)}\Big)\Big(\frac{\lambda_2^{-2} + \lambda_1^2\lambda_2^2}{2}\Big)\Big]^{\beta_r - 1}\Big\},$$

$$t_2(\lambda_1,\lambda_2) = \frac{1}{4}\sum_r^s \mu_r\Big(1 - w_1^{(r)}\Big) \Big\{ \big(\lambda_2 - \lambda_1^{-2}\lambda_2^{-3}\big)\Big[w_1^{(r)}\lambda_1^2 + \Big(1 - w_1^{(r)}\Big)\Big(\frac{\lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}}{2}\Big)\Big]^{\alpha_r - 1} - \big(\lambda_2^{-3} - \lambda_1^2\lambda_2\big)\Big[w_1^{(r)}\lambda_1^{-2} + \Big(1 - w_1^{(r)}\Big)\Big(\frac{\lambda_2^{-2} + \lambda_1^2\lambda_2^2}{2}\Big)\Big]^{\beta_r - 1}\Big\},$$

$$t_3 = 0. \tag{101}$$

Since in this loading case the calendering direction is stress-free

$$t_1(\lambda_1,\lambda_2) = 0. \tag{102}$$

Under consideration of the expression for t_1 $(101)_1$ the above condition delivers λ_1 as an implicit function of λ_2.

Thus, in the case of uniaxial loading along the calendering and transverse directions our constitutive model can be represented by the stress–stretch relations (100) and (101), respectively. To fit these relations to the experimental data we have used the least-squares method with the following objective function

$$S = \sum_{i=1}^{l}\Big[\bar{t}_1 - t_1\Big(\bar{\lambda}^{(i)}\Big)\Big]^2 + \sum_{i=1}^{m}\Big[\bar{t}_2 - t_2\Big(\lambda_1^{(i)},\bar{\lambda}_2^{(i)}\Big)\Big]^2 + \sum_{i=1}^{m}\Big[t_1\Big(\lambda_1^{(i)},\bar{\lambda}_2^{(i)}\Big)\Big]^2, \tag{103}$$

where the overbar indicates an experimental value. Herein, the first and second sums take into account errors in the approximation of the experimental data for stresses in the calendering and transverse directions, respectively, while the third sum considers the inaccuracy in the fulfillment of the condition (102). The number of data points used for the approximation of the experimental curves for the calendering and transverse directions is denoted by l and m, respectively.

The objective function (103) has been minimized with respect to the material constants μ_r, α_r, β_r and $w_1^{(r)}(r = 1,2,\ldots,s)$ and the principal stretches $\lambda_1^{(i)}(i = 1,\ldots,m)$. The power series in the strain energy function (84) has been truncated at the third term $(s = 3)$ such that the total number of unknown material constants was 12.

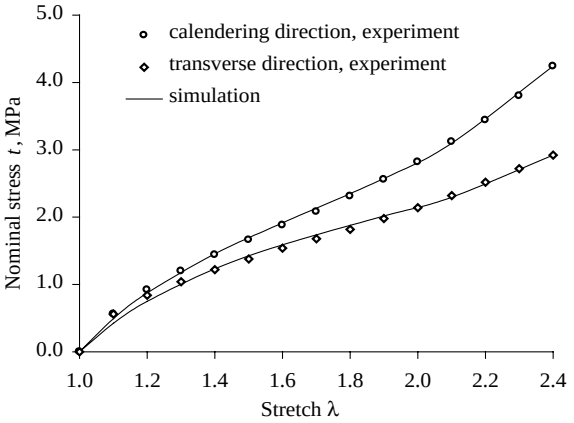

Fig. 2. Experimental data (Diani et al., 2003) on the uniaxial tension of calendered rubber versus simulation by the hyperelastic model (84) with the material constants (104).

The minimization of the objective function has been carried out with the aid of a Newton–Raphson type algorithm. The results of the fitting are illustrated in Fig. 2. A very good agreement with the experimental data is observable.

The material constants are found to be

$$\begin{aligned}
&\mu_1 = 4.387 \cdot 10^{-3}\,\text{MPa}, \quad w_1^{(1)} = 1, \quad w_2^{(1)} = w_3^{(1)} = 0, \quad \alpha_1 = 4.094, \quad \beta_1 = 2.268,\\
&\mu_2 = 4.131 \cdot 10^{-4}\,\text{MPa}, \quad w_1^{(2)} = 0, \quad w_2^{(2)} = w_3^{(2)} = 0.5, \quad \alpha_2 = 7.595, \quad \beta_2 = 8.178,\\
&\mu_3 = 5.127\,\text{MPa}, \quad w_1^{(3)} = 0.396, \quad w_2^{(3)} = w_3^{(3)} = 0.302, \quad \alpha_3 = 1.00, \quad \beta_3 = 1.00.
\end{aligned} \tag{104}$$

The relations (93) deliver further

$$\begin{aligned}
&a_{11} = 2.507\,\text{MPa}, \quad a_{12} = a_{13} = -1.253\,\text{MPa}, \quad a_{22} = a_{33} = 2.175\,\text{MPa}, \quad a_{23} = -0.922\,\text{MPa},\\
&2G_{12} = 2G_{13} = 3.584, \quad 2G_{23} = 3.097\,\text{MPa}.
\end{aligned} \tag{105}$$

Under consideration of (35)–(37) we finally obtain for the Young's moduli of the calendered rubber

$$E_1 = 5.637\ \text{MPa}, \quad E_2 = E_3 = 4.859\ \text{MPa}. \tag{106}$$

11. Conclusion

In the present paper we have proposed a class of orthotropic and transversely isotropic strain energy functions that are polyconvex and proved to be coercive. For a boundary value problem this ensures the existence of the global minimizer of the total elastic energy of the body. The proposed strain energy functions are represented by a power series with an arbitrary number terms. Each of these terms satisfies a priori the condition of the energy- and stress-free natural state such that no additional restrictions should be imposed on the associated material coefficients. These coefficients can further be evaluated on the basis of experimental data. For illustration, our hyperelastic model is thus matched to experimental data on calendered rubber sheets revealing transverse isotropy with respect to the calendering direction. A very good agreement with the experimental results has been achieved.

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